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PATTERN FORMATION IN QUANTUM ENSEMBLES

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We present a family of methods, analytical and numerical, which can describe behaviour in (non) equilibrium ensembles, both classical and quantum, especially in the complex systems, where the standard approaches cannot be applied. We demonstrate the creation of nontrivial (meta) stable states (patterns), localized, chaotic, entangled or decoherent, from basic localized modes in various collective models arising from the quantum hierarchy of Wigner-von Neumann-Moyal-Lindblad equations, which are the result of “wignerization” procedure of classical BBGKY hierarchy. We present the explicit description of internal quantum dynamics by means of exact analytical/numerical computations.

Keywords: Localization; pattern formation; multiscales; multiresolution; waveletons; (non) equilibrium ensembles.

1. Localized Modes (“continuous qudits”): Why Need We Them?

It is widely known that the currently available experimental techniques (and, apparently, those which will become available in the nearest future) in the area of quantum physics as a whole and in that of quantum computations in particular, as well as the present level of understanding of phenomenological models, outstripped the actual level of mathematical/theoretical description¹. Considering, for example, the problem of describing the realizable states (Refs. 2–7), one should not expect that planar waves and (squeezed) gaussian coherent states would be enough to characterize such complex systems as qCPU (quantum Central Processor Unit)-like devices. Complexity of the set of relevant states, including entangled (chaotic) states is still far from being clearly understood and moreover from being realizable. As a starting point for our approach let us consider the following well-known example of GKP (Gottesman, Kitaev, Preskill)⁸ scheme with DV (Discrete Variables)/qubit (with finite-dimensional code space embedded in the infinite-dimensional Hilbert space) or CV (Continuous Variables) for (optical) quantum computations, containing as a part (optical) nonlinearities, described by Kerr interaction or more general poly-

nomial Hamiltonians which are needed to realize the state preparation and provide the process of CV quantum computation⁸. It is an important example because:

(a) its classical counterpart is described by polynomial Hamiltonians;

(b) the proper qudits or building states (DV or CV) are well localized (but not well-defined mathematically, as we shall explain later).

One of the questions which motivated our approach is whether it is possible to keep (a) and at the same time improve (b). Our other motivations arise from the following general questions:

(A) How can we represent well localized and reasonable state in mathematically correct form?

(B) Is it possible to create entangled and other relevant states by means of these new building blocks?

In GKP scheme unphysical and not clearly defined mathematically logical qubit states are represented via infinite series of δ functions: $|0\rangle = \sum_{s=-\infty}^{\infty} \delta(x - 2s\sqrt{\pi})|x\rangle$, $|1\rangle = \sum_{s=-\infty}^{\infty} \delta(x - (2s+1)\sqrt{\pi})|x\rangle$ and approximated by the set of gaussian envelopes: $\langle x|0\rangle = N_0 \sum_{s=-\infty}^{+\infty} e^{-1/2(2sk\sqrt{\pi})^2} e^{-1/2(\frac{x-2s\sqrt{\pi}}{\Delta})^2}$, $\langle x|1\rangle = N_1 \sum_{s=-\infty}^{+\infty} e^{-1/2(2s+1)k\sqrt{\pi})^2} e^{-1/2(\frac{x-(2s+1)\sqrt{\pi}}{\Delta})^2}$.

Due to numerous mathematical and computational reasons, some of which are described below, such and related choices cannot be appropriate neither as a starting point on the route to the real qCPU device nor as a satisfactory theoretical description. So, it would appear that a first step in this direction is to find a reasonable extension of understanding of the quantum dynamics as a whole. One needs to sketch up the underlying ingredients of the theory (spaces of states, observables, measures, classes of smoothness, quantization set-up etc) in an attempt to provide the maximally extendable but at the same time really calculable and realizable description of the dynamics of quantum world. The general idea is rather simple: it is well known that the idea of “symmetry” is the key ingredient of any reasonable physical theory from classical (in)finite dimensional (integrable) Hamiltonian dynamics to different sub-planckian models based on strings (branes, orbifolds etc.) During the last century kinematical, dynamical and hidden symmetries played the key role in our understanding of physical process. Roughly speaking, the representation theory of underlying symmetry (classical or quantum, groups or (bi)algebras, finite or infinite dimensional, continuous or discrete) is a proper instrument for description of proper (orbital) dynamics. A starting point for us is a possible model for (continuous) “qudit” with subsequent description of the whole zoo of possible realizable (controllable) states/patterns which may be useful from the point of view of quantum experimentalists and engineers. The proper representation theory is well known as “local nonlinear harmonic analysis”, in particular case of simple underlying symmetry—affine group—aka wavelet analysis. From our point of view the advantages of such approach are as follows:

i) natural realization of localized states in any proper functional realization of (Hilbert) space of states,

ii) hidden symmetry of chosen realization of proper functional model provides the (whole) spectrum of possible states via the so-called multiresolution decomposition.

So, indeed, the hidden symmetry (non-abelian affine group in the simplest case) of the space of states via proper representation theory generates the physical spectrum and this procedure depends on the choice of the functional realization of the space of states. It explicitly demonstrates that the structure and properties of the functional realization of the space of states are the natural properties of physical world at the same level of importance as a particular choice of Hamiltonian, or the equation of motion, or the action principle (variational method). At the next step we need to consider the consequences of our choice i), ii) for the algebra of observables. In this direction one needs to mention the class of operators we are interested in to present proper description for a class of maximally generalized but reasonable class of problems. It seems that these must be pseudodifferential operators, especially if we underline that in the spirit of points i), ii) above we need to take Wigner-Weyl framework for constructing basic quantum equations of motions. It is obvious, that consideration of symbols of operators instead of operators themselves is the starting point as for the mathematical theory of pseudodifferential operators as for quantum dynamics formulated in the language of Wigner-like equations. It should be noted that in such picture we can naturally include the effects of self-interaction (“quantum non-linearity”) on the way of construction and subsequent analysis of nonlinear quantum models. So, our consideration will be in the framework of (Nonlinear) Pseudodifferential Dynamics (ΨDOD). As a result of i), ii), we’ll have:

iii) most sparse, almost diagonal, representation for a wide class of operators included in the set-up of the whole problems.

It’s possible by using the so-called Fast Wavelet Transform representation for algebra of observables.

Then points i)–iii) provide us by

iv) natural (non-perturbative) multiscale decomposition for all dynamical quantities, as states as observables.

The simplest case we will have, obviously, in Wigner-Weyl representation. Existence of such internal multiscales with different dynamics at each scale and transitions, interactions, and intermittency between scales demonstrates that quantum mechanics, despite its linear structure, is really a serious part of physics from the mathematical point of view. It seems, that well-known underlying quantum complexity is a result of transition by means of (still rather unclear) procedure of quantization from complexity related to nonlinearity of classical counterpart to the rich pseudodifferential (more exactly, microlocal) structure on the quantum side.

We divide all possible configurations related to possible solutions of our quantum equation of motion (Wigner-like equations, mostly) into two classes:

(**a**) standard solutions; (**b**) controllable solutions (solutions with prescribed qualitative type of behaviour).

Anyway, the whole zoo of solutions consists of possible patterns, including very important ones from the point of view of underlying physics:

v) localized modes (basis modes, eigenmodes) and constructed from them chaotic or entangled, decoherent (if we change Wigner equation for (master) Lindblad one) patterns.

It should be noted that these bases modes are nonlinear in contrast with usual ones because they come from (non) abelian generic group while the usual Fourier (commutative) analysis starts from $U(1)$ abelian modes (plane waves). They are really “eigenmodes” but in sense of decomposition of representation of the underlying hidden symmetry group which generates the multiresolution decomposition. The set of patterns is built from these modes by means of variational procedures more or less standard in mathematical physics. It allows to control the convergence from one side but, what is more important,

vi) to consider the problem of the control of patterns (types of behaviour) on the level of reduced (variational) algebraical equations.

We need to mention that it is possible to change the simplest generic group of hidden internal symmetry from the affine (translations and dilations) to much more general, but, in any case, this generic symmetry will produce the proper natural high localized eigenmodes, as well as the decomposition of the functional realization of space of states into the proper orbits; and all that allows to compute dynamical consequence of this procedure, i.e. pattern formation, and, as a result, to classify the whole spectrum of proper states.

For practical reasons controllable patterns (with prescribed behaviour) are the most useful. We mention the so-called wavelet-like pattern which we regard as the most important one. We use the following allusion in the space of words:

$$\{\text{wavelet}\} := \{\text{soliton}\} \sqcup \{\text{wavelet}\}$$

It means:

vii) wavelet \approx (meta)stable localized (controllable) pattern

To summarize, the approach described below allows one

viii) to solve wide classes of general ΨDOD problems, including generic for quantum physics Wigner-like equations, and

ix) to present the analytical/numerical realization for physically interesting patterns.

We would like to emphasize the effectiveness of numerical realization of this program (minimal complexity of calculations) as additional advantage. So, items i)-ix) point out all main features of our approach, Refs. 2–7.

2. Motivations

2.1. Class of Models

Here we describe a class of problems which can be analysed by methods described in Introduction. We start from individual dynamics and finish by (non)-equilibrium ensembles. All models belong to the ΨDOD class and can be described by finite or

infinite (named hierarchies in such cases) system of ΨDOD equations:

- a). Individual classical/quantum mechanics (cM/qM): linear/nonlinear; $\{cM\} \subset \{qM\}$, $*$ - quantized for the class of polynomial Hamiltonians $H(p, q, t) = \sum_{i,j} a_{ij}(t) p^i q^j$.
- b). QFT-like models in framework of the second quantization (dynamics in Fock spaces).
- c). Classical (non) equilibrium ensembles via BBGKY Hierarchy (with reductions to different forms of Vlasov-Maxwell/Poisson equations).
- d). Wignerization of a): Wigner-Moyal-Weyl-von Neumann-Lindblad.
- e). Wignerization of c): Quantum (Non) Equilibrium Ensembles.

Important remarks: points a)-e) are considered in ΨDO picture of (Non)Linear ΨDO Dynamics (surely, all $qM \subset \Psi DOD$); dynamical variables/observables are the symbols of operators or functions; in case of ensembles, the main set of dynamical variables consists of partitions (n-particle partition functions).

2.2. Effects we are interested in

- (i) Hierarchy of internal/hidden scales (time, space, phase space).
- (ii) Non-perturbative multiscales: from slow to fast contributions, from the coarser to the finer level of resolution/decomposition.
- (iii) Coexistence of hierarchy of multiscale dynamics with transitions between scales.
- (iv) Realization of the key features of the complex quantum world such as the existence of chaotic and/or entangled states with possible destruction in “open/dissipative” regimes due to interactions with quantum/classical environment and transition to decoherent states.

At this level we may interpret the effect of mysterious entanglement or “quantum interaction” as a result of simple interscale interaction or intermittency (with allusion to hydrodynamics), i.e. the mixing of orbits generated by multiresolution representation of hidden underlying symmetry. Surely, the concrete realization of such a symmetry is a natural physical property of the physical model as well as the space of representation and its proper functional realization. So, instantaneous interactions (or transmission of “quantum bits” or “teleportation”) materialize not in the physical space-time variety but in the space of representation of hidden symmetry along the orbits/scales constructed by proper representations. Dynamical/kinematical principles of usual space-time varieties, definitely, do not cover kinematics of internal quantum space of state or, in more weak formulation, we still have not such explicit relations.

One additional important comment: as usual in modern physics, we have the hierarchy of underlying symmetries; so our internal symmetry of functional realization of space of states is really not more than kinematical, because much more rich algebraic structure, related to operator Cuntz algebra and quantum groups, is hidden inside. The proper representations can generate much more interesting effects

than ones described above. We will consider it elsewhere but mention here only how it can be realized by the existing functorial maps between proper categories:

$\{\text{QMF}\} \longrightarrow \text{Loop groups} \longrightarrow \text{Cuntz operator algebra} \longrightarrow \text{Quantum Group structure}$, where $\{\text{QMF}\}$ are the so-called quadratic mirror filters generating the realization of multiresolution decomposition/representation in any functional space; loop group is well known in many areas of physics, e.g. soliton theory, strings etc, roughly speaking, its algebra coincides with Virasoro algebra; Cuntz operator algebra is universal C^* algebra generated by N elements with two relations between them; Quantum group structure (bialgebra, Hopf algebra, etc) is well known in many areas because of its universality. It should be noted the appearance of natural Fock structure inside this functorial sequence above with the creation operator realized as some generalization of Cuntz-Toeplitz isometries. Surely, all that can open a new vision of old problems and bring new possibilities.

We finish this part by the following qualitative definitions of key objects (patterns). Their description and understanding in different physical models is our main goal in this direction.

- By localized states (localized modes) we mean the building blocks for solutions or generating modes which are localized in maximally small region of the phase (as in c - as in q -case) space.
- By an entangled/chaotic pattern we mean some solution (or asymptotics of solution) which has random-like distributed energy (or information) spectrum in a full domain of definition. In quantum case we need to consider additional entangled-like patterns, roughly speaking, which cannot be separated into pieces of subsystems.
- By a localized pattern (wavelet) we mean (asymptotically) (meta) stable solution localized in a relatively small region of the whole phase space (or a domain of definition). In this case the energy is distributed during some time (sufficiently large) between only a few localized modes (from point 1). We believe it to be a good model for plasma in a fusion state (energy confinement) or a model for quantum continuous “qubit” or a result of the decoherence process in open quantum system when the full entangled state degenerates into localized (quasiclassical) pattern.

2.3. *Methods*

- (i) Representation theory of internal/hidden/underlying symmetry, Kinematical, Dynamical, Hidden.
- (ii) Arena (space of representation): proper functional realization of (Hilbert) space of states.
- (iii) Harmonic analysis on (non)abelian group of internal symmetry. Local/Nonlinear (non-abelian) Harmonic Analysis (e.g. wavelet/gabor etc. analysis) instead of linear non-localized $U(1)$ Fourier analysis. Multiresolution (multiscale) representation. Dynamics on proper orbit/scale (inside the whole hi-

erarchy of multiscales) in functional space. The key ingredients are the appearance of multiscales (orbits) and the existence of high-localized natural (eigen)modes⁹.

- (iv) Variational formulation (control of convergence, reductions to algebraic systems, control of type of behaviour).

3. Set-up/Formulation

Let us consider the following generic Ψ DOD dynamical problem

$$L^j\{Op^i\}\Psi = 0,$$

described by a finite or infinite number of equations which include general classes of operators Op^i such as differential, integral, pseudodifferential etc

Surely, all Wigner-like equations/hierarchies are inside.

The main objects are:

- (i) (Hilbert) space of states, $H = \{\Psi\}$, with a proper functional realization, e.g.,: L^2 , Sobolev, Schwartz, C^0 , C^k , ... C^∞ , ...; Definitely, $L^2(R^2)$, $L^2(S^2)$, $L^2(S^1 \times S^1)$, $L^2(S^1 \times S^1 \times Z_n)$ are different objects proper for different physics inside.
- (ii) Class of smoothness. The proper choice determines natural consideration of dynamics with/without Chaos/Fractality property.
- (iii) Decompositions

$$\Psi \approx \sum_i a_i e^i$$

via high-localized bases (wavelet families, generic wavelet packets etc), frames, atomic decomposition (Fig. 1) with the following main properties: (exp) control of convergence, maximal rate of convergence for any Ψ in any H .

- (iv) Observables/Operators (ODO, PDO, Ψ DO, SIO,..., Microlocal analysis of Kashiwara-Shapira (with change from functions to sheafs)) satisfy the main property – the matrix representation in localized bases

$$\langle \Psi | Op^i | \Psi \rangle$$

is maximum sparse:

$$\begin{pmatrix} D_{11} & 0 & 0 & \dots \\ 0 & D_{22} & 0 & \dots \\ 0 & 0 & D_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This almost diagonal structure is provided by the so-called Fast Wavelet Transform.

- (v) Measures: multifractal wavelet measures $\{\mu_i\}$ together with the class of smoothness are very important for analysis of complicated analytical behaviour.

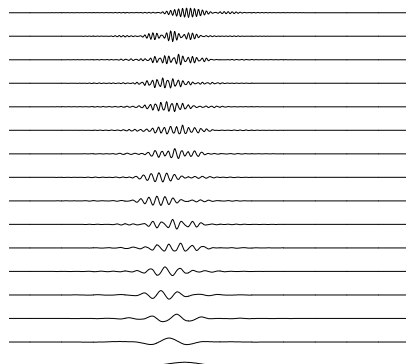


Fig. 1. Localized modes.

- (vi) Variational/Projection methods, from Galerkin to Rabinowitz minimax, Floer (in symplectic case of Arnold-Weinstein curves with preservation of Poisson/symplectic structures). Main advantages are the reduction to algebraic systems, which provides a tool for the smart subsequent control of behaviour and control of convergence.
 - (vii) Multiresolution or multiscale decomposition, *MRA* (or wavelet microscope) consists of the understanding and choosing of
 - 1). (internal) symmetry structure, e.g., affine group = {translations, dilations} or many others; construction of
 - 2). representation/action of this symmetry on $H = \{\Psi\}$.
 As a result of such hidden coherence together with using point vi) we'll have:
 - a). LOCALIZED BASES b). EXACT MULTISCALE DECOMPOSITION with the best convergence properties and real evaluation of the rate of convergence via proper "multi-norms".
- Figures 2, 3, 5, 6 demonstrate *MRA* decompositions for one- and multi-kicks while Figures 4 and 7 present the same for the case of the generic simple fractal model, Riemann-Weierstrass function [9].
- (viii) Effectiveness of proper numerics: CPU-time, HDD-space, minimal complexity of algorithms, and (Shannon) entropy of calculations are provided by points i)-vii) above.
 - (ix) Quantization via $*$ star product or Deformation Quantization.

The corresponding class of individual Hamiltonians has the form

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + U(\hat{p}, \hat{q}), \quad (1)$$

where $U(\hat{p}, \hat{q})$ is an arbitrary polynomial function on \hat{p}, \hat{q} , and plays the key role in many areas of physics¹. Our starting point is the general point of view of a deformation quantization approach at least on the naive Moyal/Weyl/Wigner level. The main point of such approach is based on ideas from¹, which allow to consider the algebras of quantum observables as the deformations of commutative algebras of classical observables (functions). So, if we have the classical counterpart of Hamiltonian (1) as a model for classical dynamics and the Poisson manifold M (or symplectic manifold or Lie coalgebra, etc) as the corresponding phase space, then for quantum calculations we need first of all to find an associative (but non-commutative) star product $*$ on the space of formal power series in \hbar with coefficients in the space of smooth functions on M such that $f * g = fg + \hbar\{f, g\} + \sum_{n \geq 2} \hbar^n B_n(f, g)$, where $\{f, g\}$ is the Poisson brackets, B_n are bidifferential operators. In this paper we consider the calculations of the Wigner functions $W(p, q, t)$ (WF) corresponding to the classical polynomial Hamiltonian $H(p, q, t)$ as the solution of the Wigner-von Neumann equation¹:

$$i\hbar \frac{\partial}{\partial t} W = H * W - W * H \quad (2)$$

and related Wigner-like equations for different ensembles. According to the Weyl transform, a quantum state (wave function or density operator ρ) corresponds to the Wigner function, which is the analogue in some sense of classical phase-space distribution¹. Wigner equation (2) is a result of the Weyl transform or “wignerization” of von Neumann equation for density matrix.

Finally, such Variational-Multiscale approach based on points i)-ix) provides us by the full ZOO of PATTERNS: LOCALIZED, CHAOTIC/ENTANGLED, etc.

In next Sections we will consider details for important cases of Wigner-like equations.

We present the explicit analytical construction for solutions of c- and q-hierarchies and their important reductions starting from quantization of c-BBGKY (Born-Bogolyubov-Green-Kirkwood-Yvon) hierarchy, which is based on tensor algebra extensions of multiresolution representation for states and observables and variational formulation. We give explicit representation for hierarchy of n-particle reduced distribution functions in the base of high-localized generalized coherent (regarding underlying generic symmetry (affine group in the simplest case)) states given by polynomial tensor algebra of our basis functions (wavelet families, wavelet packets), which takes into account contributions from all underlying hidden multiscales from the coarsest scale of resolution to the finest one to provide full information about (quantum) dynamical process. The difference between classical and quantum case is concentrated in the structure of the set of operators included in the set-up and, surely, depends on the method of quantization. But, in the naive Wigner-Weyl approach for quantum case the symbols of operators play the same role as usual functions in classical case. In some sense, our approach for ensembles (hierarchies) resembles Bogolyubov’s one and related approaches but we don’t use any perturba-

tion technique (like virial expansion) or linearization procedures. Most important, that numerical modeling in all cases shows the creation of different internal (coherent) structures from localized modes, which are related to stable (equilibrium) or unstable type of behaviour and corresponding pattern (waveleons) formation.

4. BBGKY/Wigner Ensembles: from c - to q -cases

We start from set-up for kinetic BBGKY hierarchy as c -counterpart of proper q -hierarchy. Let M be the phase space of ensemble of N particles ($\dim M = 6N$) with coordinates $x_i = (q_i, p_i)$, $i = 1, \dots, N$, $q_i = (q_i^1, q_i^2, q_i^3) \in R^3$, $p_i = (p_i^1, p_i^2, p_i^3) \in R^3$, $q = (q_1, \dots, q_N) \in R^{3N}$. Individual and collective measures are: $\mu_i = dx_i = dq_i dp_i$, $\mu = \prod_{i=1}^N \mu_i$ while distribution function $D_N(x_1, \dots, x_N; t)$ satisfies Liouville equation of motion for ensemble with Hamiltonian H_N and normalization constraint. Our constructions can be applied to the following general Hamiltonians:

$$H_N = \sum_{i=1}^N \left(\frac{p_i^2}{2m} + U_i(q) \right) + \sum_{1 \leq i < j \leq N} U_{ij}(q_i, q_j), \quad (3)$$

where potentials $U_i(q) = U_i(q_1, \dots, q_N)$ and $U_{ij}(q_i, q_j)$ are not more than rational functions on coordinates. Let L_s and L_{ij} be the standard Liouvillean operators and

$$F_s(x_1, \dots, x_s; t) = V^s \int D_N(x_1, \dots, x_N; t) \prod_{s+1 \leq i \leq N} \mu_i \quad (4)$$

be the hierarchy of reduced distribution functions, then, after standard manipulations, we arrive to c -BBGKY hierarchy:

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{v} \int d\mu_{s+1} \sum_{i=1}^s L_{i,s+1} F_{s+1}. \quad (5)$$

So, the proper dynamical formulation is reduced to the (infinite) set of equations for correlators/partition functions. Then by using physical motivated reductions or/and during the corresponding cut-off procedure we obtain, instead of linear and pseudodifferential (in general case) equations, their finite-dimensional but nonlinear approximations with the polynomial type of nonlinearities (more exactly, multilinearity). To move from c - to q -case, let us start from the second quantized representation for an algebra of observables $A = (A_0, A_1, \dots, A_s, \dots)$ in the standard form $A = A_0 + \int dx_1 \Psi^+(x_1) A_1 \Psi(x_1) + \dots + (s!)^{-1} \int dx_1 \dots dx_s \Psi^+(x_1) \dots \Psi^+(x_s) A_s \Psi(x_s) \dots \Psi(x_1) + \dots$ N -particle Wigner functions

$$W_s(x_1, \dots, x_s) = \int dk_1 \dots dk_s \exp\left(-i \sum_{i=1}^s k_i p_i\right) \text{Tr} \rho \Psi^+\left(q_1 - \frac{1}{2} \hbar k_1\right) \dots \quad (6)$$

$$\Psi^+\left(q_s - \frac{1}{2} \hbar k_s\right) \Psi\left(q_s + \frac{1}{2} \hbar k_s\right) \dots \Psi\left(q_1 + \frac{1}{2} \hbar k_s\right)$$

allow us to consider them as some quasiprobabilities and provide useful bridge between c- and q-cases:

$$\langle A \rangle = \text{Tr} \rho A = \sum_{s=0}^{\infty} (s!)^{-1} \int \prod_{i=1}^s d\mu_i A_s(x_1, \dots, x_s) W_s(x_1, \dots, x_s). \quad (7)$$

The full description for quantum ensemble can be done by the whole hierarchy of functions (symbols): $W = \{W_s(x_1, \dots, x_s), s = 0, 1, 2, \dots\}$. So, we may consider the following q-hierarchy as the result of “wignerization” procedure for c-BBGKY one:

$$\begin{aligned} \partial_t W_s(t, x_1, \dots, x_s) = & \sum_{j=1}^s L_j^0 W_s(x_1, \dots, x_s) + \sum_{j < n} \sum_{n=1}^s L_{j,n} W_s(x_1, \dots, x_s) \\ & + \sum_{j=1}^s \int dx_{s+1} \delta(k_{s+1}) L_{j,s+1} W_{s+1}(x_1, \dots, x_{s+1}), \end{aligned} \quad (8)$$

$$\begin{aligned} L_j^0 = & -\left(\frac{i}{m}\right) k_j p_j, \quad L_{j,n} = (i\hbar)^{-1} \int d\ell \tilde{V}_l \left[\exp\left(-\frac{1}{2}\hbar\ell\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_n}\right)\right) - \right. \\ & \left. \exp\left(\frac{1}{2}\hbar\ell\left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_n}\right)\right) \right] \exp\left(-\ell\left(\frac{\partial}{\partial k_j} - \frac{\partial}{\partial k_n}\right)\right). \end{aligned} \quad (9)$$

In quantum statistics the ensemble properties are described by the density operator

$$\rho(t) = \sum_i w_i |\Psi_i(t)\rangle \langle \Psi_i(t)|, \quad \sum_i w_i = 1. \quad (10)$$

After Weyl transform we have the following decomposition via partial Wigner functions $W_i(p, q, t)$ for the whole ensemble Wigner function:

$$W(p, q, t) = \sum_i w_i W_i(p, q, t), \quad (11)$$

where the partial Wigner functions

$$W_n(q, p, t) \equiv \frac{1}{2\pi\hbar} \int d\xi \exp\left(-\frac{i}{\hbar} p\xi\right) \Psi_n^*(q - \frac{1}{2}\xi, t) \Psi_n(q + \frac{1}{2}\xi, t) \quad (12)$$

are solutions of proper Wigner equations:

$$\frac{\partial W_n}{\partial t} = -\frac{p}{m} \frac{\partial W_n}{\partial q} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\hbar/2)^{2\ell}}{(2\ell+1)!} \frac{\partial^{2\ell+1} U_n(q)}{\partial q^{2\ell+1}} \frac{\partial^{2\ell+1} W_n}{\partial p^{2\ell+1}}. \quad (13)$$

Our approach, presented below, in some sense has allusion on the analysis of the following standard simple model considered in Ref. 1. Let us consider model of interaction of nonresonant atom with quantized electromagnetic field: $\hat{H} =$

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$\frac{\hat{p}_x^2}{2m} + U(\hat{x})$, $U(\hat{x}) = U_0(z, t)g(\hat{x})\hat{a}^+\hat{a}$, where potential U depends on creation/annihilation operators and some polynomial on \hat{x} operator function (or approximation) $g(\hat{x})$. It is possible to solve Schroedinger equation $i\hbar d|\Psi\rangle/dt = \hat{H}|\Psi\rangle$ by the simple ansatz

$$|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} w_n \int dx \Psi_n(x, t) |x\rangle \otimes |n\rangle, \quad (14)$$

which leads to the hierarchy of analogous equations with potentials created by n -particle Fock subspaces

$$i\hbar \frac{\partial \Psi_n(x, t)}{\partial t} = \left\{ \frac{\hat{p}_x^2}{2m} + U_0(t)g(x)n \right\} \Psi_n(x, t), \quad (15)$$

where $\Psi_n(x, t)$ is the probability amplitude of finding the atom at the time t at the position x and the field in the n Fock state. Instead of this, we may apply the Wigner approach starting with proper full density matrix $|\Psi(t)\rangle \langle \Psi(t)|$:

$$\hat{\rho} = \sum_{n', n''} w_{n'} w_{n''}^* \int dx' \int dx'' \Psi_{n'}(x', t) \Psi_{n''}^*(x'', t) |x'\rangle \langle x''| \otimes |n'\rangle \langle n''|. \quad (16)$$

Standard reduction gives pure atomic density matrix

$$\begin{aligned} \hat{\rho}_a &\equiv \int_{n=0}^{\infty} \langle n | \hat{\rho} | n \rangle = \\ &\sum |w_n|^2 \int dx' \int dx'' \Psi_n(x', t) \Psi_n^*(x'', t) |x'\rangle \langle x''|. \end{aligned} \quad (17)$$

Then we have incoherent superposition

$$W(x, p, t) = \sum_{n=0}^{\infty} |w_n|^2 W_n(x, p, t) \quad (18)$$

of the atomic Wigner functions (12) corresponding to the atom motion in the potential $U_n(x)$ (which is not more than polynomial in x) generated by n -level Fock state. They are solutions of proper Wigner equations (13). The next case describes the important decoherence process. Let us have collective and environment subsystems with their own Hilbert spaces $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_e$. Relevant dynamics is described by three parts including interaction $H = H_c \otimes I_e + I_c \otimes H_e + H_{int}$. For analysis, we can choose Lindblad master equation¹

$$\dot{\rho} = \frac{1}{i\hbar} [H, \rho] - \sum_n \gamma_n (L_n^+ L_n \rho + \rho L_n^+ L_n - 2L_n \rho L_n^+), \quad (19)$$

which preserves the positivity of density matrix and it is Markovian but it is not general form of exact master equation. Other choice is Wigner transform of master equation:

$$\begin{aligned} \dot{W} &= \{H, W\}_{PB} + \\ &\sum_{n \geq 1} \frac{\hbar^{2n} (-1)^n}{2^{2n} (2n+1)!} \partial_q^{2n+1} U(q) \partial_p^{2n+1} W(q, p) + 2\gamma \partial_p p W + D \partial_p^2 W, \end{aligned} \quad (20)$$

and it is more preferable for us. In the next Section we consider the variational-wavelet approach Refs. 2–7 for the solution of all these Wigner-like equations (2), (8), (13), (20) for the case of an arbitrary polynomial $U(q, p)$, which corresponds to a finite number of terms in the series expansion in (13), (20) or to proper finite order of \hbar . Analogous approach can be applied to classical counterpart (5) also. Roughly speaking, wavelet analysis⁹ is some set of mathematical methods, which gives the possibility to take into account high-localized states, control convergence of any type of expansions and gives maximum sparse forms for the general type of operators in such localized bases. These bases are the natural generalization of standard coherent, squeezed, thermal squeezed states¹, which correspond to quadratic systems (pure linear dynamics) with Gaussian Wigner functions. The representations of underlying symmetry group (affine group in the simplest case) on the proper functional space of states generate the exact multiscale expansions which allow to control contributions to the final result from each scale of resolution from the whole underlying infinite scale of spaces.

5. Variational Multiresolution Representation

5.1. Multiscale Decomposition for Space of States: Functional Realization and Metric Structure

We obtain our multiscale/multiresolution representations for solutions of Wigner-like equations via a variational-wavelet approach. We represent the solutions as decomposition into localized eigenmodes (regarding action of affine group, i.e. hidden symmetry of the underlying functional space of states) related to the hidden underlying set of scales:

$$W_n(t, q, p) = \bigoplus_{i=i_c}^{\infty} W_n^i(t, q, p), \quad (21)$$

where value i_c corresponds to the coarsest level of resolution c or to the internal scale with the number c in the full multiresolution decomposition (MRA) of the underlying functional space (L^2 , e.g.) corresponding to the problem under consideration⁹:

$$V_c \subset V_{c+1} \subset V_{c+2} \subset \dots \quad (22)$$

and $p = (p_1, p_2, \dots)$, $q = (q_1, q_2, \dots)$, $x_i = (p_1, q_1, \dots, p_i, q_i)$ are coordinates in phase space. In the following we may consider as fixed as variable numbers of particles. We introduce the Fock-like space structure (in addition to the standard one, if we consider second-quantized case) on the whole space of internal hidden scales

$$H = \bigoplus_i \bigotimes_n H_i^n \quad (23)$$

for the set of n -partial Wigner functions (states):

$$W^i = \{W_0^i, W_1^i(x_1; t), \dots, W_N^i(x_1, \dots, x_N; t), \dots\}, \quad (24)$$

where $W_p(x_1, \dots, x_p; t) \in H^p$, $H^0 = C$, $H^p = L^2(R^{6p})$ (or any different proper functional space), with the natural Fock space like norm:

$$(W, W) = W_0^2 + \sum_i \int W_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^i \mu_\ell. \quad (25)$$

First of all, we consider $W = W(t)$ as a function of time only, $W \in L^2(R)$, via multiresolution decomposition which naturally and efficiently introduces the infinite sequence of the underlying hidden scales ⁹. We have the contribution to the final result from each scale of resolution from the whole infinite scale of spaces (22). The closed subspace V_j ($j \in \mathbf{Z}$) corresponds to the level j of resolution, or to the scale j and satisfies the following properties: let D_j be the orthonormal complement of V_j with respect to V_{j+1} : $V_{j+1} = V_j \oplus D_j$. Then we have the following decomposition:

$$\{W(t)\} = \bigoplus_{-\infty < j < \infty} D_j = \overline{V_c \bigoplus_{j=0}^{\infty} D_j}, \quad (26)$$

in case when V_c is the coarsest scale of resolution. The subgroup of translations generates a basis for the fixed scale number: $\text{span}_{k \in \mathbf{Z}} \{2^{j/2} \Psi(2^j t - k)\} = D_j$. The whole basis is generated by action of the full affine group:

$$\text{span}_{k \in \mathbf{Z}, j \in \mathbf{Z}} \{2^{j/2} \Psi(2^j t - k)\} = \text{span}_{k, j \in \mathbf{Z}} \{\Psi_{j, k}\} = \{W(t)\}. \quad (27)$$

5.2. Tensor Product Structure

Let sequence $\{V_j^t\}, V_j^t \subset L^2(R)$ correspond to multiresolution analysis on time axis and $\{V_j^{x_i}\}, V_j^{x_i} \subset L^2(R)$ correspond to multiresolution analysis for coordinate x_i , then $V_j^{n+1} = V_j^{x_1} \otimes \dots \otimes V_j^{x_n} \otimes V_j^t$ corresponds to multiresolution analysis for n -particle distribution function $W_n(x_1, \dots, x_n; t)$. E.g., for $n = 2$:

$$V_0^2 = \{f : f(x_1, x_2) = \sum_{k_1, k_2} a_{k_1, k_2} \phi^2(x_1 - k_1, x_2 - k_2), \quad a_{k_1, k_2} \in \ell^2(\mathbf{Z}^2)\}, \quad (28)$$

where $\phi^2(x_1, x_2) = \phi^1(x_1) \phi^1(x_2) = \phi^1 \otimes \phi^1(x_1, x_2)$, and $\phi^i(x_i) \equiv \phi(x_i)$ form a multiresolution basis corresponding to $\{V_j^{x_i}\}$. If $\{\phi^1(x_1 - \ell)\}$, $\ell \in \mathbf{Z}$ form an orthonormal set, then $\phi^2(x_1 - k_1, x_2 - k_2)$ form an orthonormal basis for V_0^2 . Action of affine group provides us by multiresolution representation of $L^2(R^2)$. After introducing detail spaces D_j^2 , we have, e.g. $V_1^2 = V_0^2 \oplus D_0^2$. Then 3-component basis for D_0^2 is generated by translations of three functions

$$\Psi_1^2 = \phi^1(x_1) \otimes \Psi^2(x_2), \quad \Psi_2^2 = \Psi^1(x_1) \otimes \phi^2(x_2), \quad \Psi_3^2 = \Psi^1(x_1) \otimes \Psi^2(x_2). \quad (29)$$

Also, we may use the rectangle lattice of scales and one-dimensional wavelet decomposition :

$$f(x_1, x_2) = \sum_{i, \ell; j, k} \langle f, \Psi_{i, \ell} \otimes \Psi_{j, k} \rangle \Psi_{j, \ell} \otimes \Psi_{j, k}(x_1, x_2), \quad (30)$$

where bases functions $\Psi_{i, \ell} \otimes \Psi_{j, k}$ depend on two scales 2^{-i} and 2^{-j} .

After construction the multidimensional bases we obtain our multiscale/multi-resolution representations for observables (symbols), states, partitions via the variational approaches in Refs. 2–7 as for c-BBGKY as for its quantum counterpart and related reductions but before we need to construct reasonable multiscale decomposition for all operators included in the set-up.

5.3. FWT Decomposition for Observables

One of the key point of wavelet analysis approach, the so called Fast Wavelet Transform (FWT) ⁹, demonstrates that for the large classes of operators the wavelet-like functions are best approximation for true eigenvectors and the corresponding matrices are almost diagonal. So, powerful FWT provides the maximum sparse form for different classes of operators ⁹. Let us denote our (integral/differential) operator from equations under consideration as T ($L^2(R^n) \rightarrow L^2(R^n)$) and its kernel as K . We have the following representation:

$$\langle Tf, g \rangle = \int \int K(x, y) f(y) g(x) dx dy. \quad (31)$$

In case when f and g are wavelets $\varphi_{j,k} = 2^{j/2} \varphi(2^j x - k)$, (21) provides the standard representation for operator T . Let us consider multiresolution representation $\dots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \dots$. The basis in each V_j is $\varphi_{j,k}(x)$, where indices k, j represent translations and scaling respectively. Let $P_j : L^2(R^n) \rightarrow V_j$ ($j \in Z$) be projection operators on the subspace V_j corresponding to level j of resolution: $(P_j f)(x) = \sum_k \langle f, \varphi_{j,k} \rangle \varphi_{j,k}(x)$. Let $Q_j = P_{j-1} - P_j$ be the projection operator on the subspace D_j ($V_{j-1} = V_j \oplus D_j$), then we have the following representation of operator T which takes into account contributions from each level of resolution from different scales starting with the coarsest and ending to the finest scales ⁹:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j). \quad (32)$$

We need to remember that this is a result of presence of affine group inside this construction. The non-standard form of operator representation is a representation of operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j \in Z}$, acting on the subspaces V_j and D_j : $A_j : D_j \rightarrow D_j$, $B_j : V_j \rightarrow D_j$, $\Gamma_j : D_j \rightarrow V_j$, where operators $\{A_j, B_j, \Gamma_j\}_{j \in Z}$ are defined as $A_j = Q_j T Q_j$, $B_j = Q_j T P_j$, $\Gamma_j = P_j T Q_j$. The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}, \quad (33)$$

where $T_j = P_j T P_j$ and T_j acts on $V_j : V_j \rightarrow V_j$. So, it is possible to provide the following “sparse” action of operator T_j on elements f of functional realization of our space of states H :

$$(T_j f)(x) = \sum_{k \in Z} \left(2^{-j} \sum_{\ell} r_{\ell} f_{j,k-\ell} \right) \varphi_{j,k}(x), \quad (34)$$

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in the wavelet basis $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x - k)$, where

$$f_{j,k-1} = 2^{-j/2} \int f(x)\varphi(2^{-j}x - k + \ell)dx \quad (35)$$

are wavelet coefficients and r_ℓ are the roots of some additional linear system of equations related to the “type of localization”⁹. So, we have the simple linear parametrization of matrix representation of our operators in localized wavelet bases and of the action of this operator on arbitrary vector/state in proper functional space.

5.4. Variational Approach

Now, after preliminary work with (functional) spaces, states and operators, we may apply our variational approach from [2]-[7]. Let L be an arbitrary (non)linear differential/integral operator with matrix dimension d (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes n})$: $\Psi \equiv \Psi(t, x_1, x_2, \dots) = (\Psi^1(t, x_1, x_2, \dots), \dots, \Psi^d(t, x_1, x_2, \dots))$, $x_i \in \Omega \subset \mathbf{R}^6$, n is the number of particles:

$$L\Psi \equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \quad (36)$$

$$\begin{aligned} Q &\equiv Q_{d_0, d_1, d_2, \dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) \\ &= \sum_{i_0, i_1, i_2, \dots=1}^{d_0, d_1, d_2, \dots} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots) \left(\frac{\partial}{\partial t}\right)^{i_0} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \left(\frac{\partial}{\partial x_2}\right)^{i_2} \dots \int \mu_k. \end{aligned}$$

Let us consider now the N mode approximation for the solution as the following ansatz:

$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0, i_1, i_2, \dots=1}^N a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots(t, x_1, x_2, \dots) \quad (37)$$

We will determine the expansion coefficients from the following conditions (related to proper choosing of variational approach):

$$\ell_{k_0, k_1, k_2, \dots}^N \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) dt dx_1 dx_2 \dots = 0. \quad (38)$$

Thus, we have exactly dN^n algebraical equations for dN^n unknowns $a_{i_0, i_1, \dots}$. This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second one. It allows to unify the multiresolution expansion with variational construction in Refs. 2-7.

As a result, the solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the generic operator L) and the rest are linear problems related to the computation of the coefficients of reduced algebraic equations. It is also related to the choice of

exact measure of localization (including class of smoothness) which are proper for our set-up. These coefficients can be found by some functional/algebraic methods by using the compactly supported wavelet basis functions or any other wavelet families⁹. As a result the solution of the equations/hierarchies from Section 4, as in c- as in q-region, has the following multiscale or multiresolution decomposition via nonlinear high-localized eigenmodes

$$\begin{aligned} W(t, x_1, x_2, \dots) &= \sum_{(i,j) \in Z^2} a_{ij} U^i \otimes V^j(t, x_1, \dots), \\ V^j(t) &= V_N^{j,slow}(t) + \sum_{l \geq N} V_l^j(\omega_l t), \quad \omega_l \sim 2^l, \end{aligned} \quad (39)$$

$$U^i(x_s) = U_M^{i,slow}(x_s) + \sum_{m \geq M} U_m^i(k_m^s x_s), \quad k_m^s \sim 2^m,$$

which corresponds to the full multiresolution expansion in all underlying time/space scales. The formulae (39) give the expansion into a slow part and fast oscillating parts for arbitrary N, M . So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process. In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. Formulae (39) do not use perturbation techniques or linearization procedures. Numerical calculations are based on compactly supported wavelets and wavelet packets and on evaluation of the accuracy on the level N of the corresponding cut-off of the full system regarding norm (25): $\|W^{N+1} - W^N\| \leq \varepsilon$.

6. Modeling of Patterns

To summarize, the key points are:

1. The ansatz-oriented choice of the (multidimensional) bases related to some polynomial tensor algebra.
2. The choice of proper variational principle. A few projection/ Galerkin-like principles for constructing (weak) solutions can be considered. The advantages of formulations related to biorthogonal (wavelet) decomposition should be noted.
3. The choice of bases functions in the scale spaces D_j from wavelet zoo. They correspond to high-localized (nonlinear) excitations, nontrivial local (stable) distributions/fluctuations or “continuous qudits”. Besides fast convergence properties it should be noted minimal complexity of all underlying calculations, especially in case of choice of wavelet packets which minimize Shannon entropy.
4. Operator representations providing maximum sparse representations for arbitrary (pseudo) differential/ integral operators df/dx , $d^n f/dx^n$, $\int T(x, y)f(y)dy$, etc.

5. (Multi)linearization. Besides the variation approach we can consider also a different method to deal with (polynomial) nonlinearities: para-products-like decompositions.

To classify the qualitative behaviour we apply standard methods from general control theory or really use the control. We will start from a priori unknown coefficients, the exact values of which will subsequently be recovered. Roughly speaking, we will fix only class of nonlinearity (polynomial in our case) which covers a broad variety of examples of possible truncation of the systems. As a simple model we choose band-triangular non-sparse matrices (a_{ij}) . These matrices provide tensor structure of bases in (extended) phase space and are generated by the roots of the reduced variational (Galerkin-like) systems. As a second step we need to restore the coefficients from these matrices by which we may classify the types of behaviour. We start with the localized mode, which is a base mode/eigenfunction, which was constructed as a tensor product of the two base functions. Fig. 8, 11 below demonstrate the result of summation of series (39) up to value of the dilation/scale parameter equal to four and six, respectively. It's done in the bases of symmlets [9] with the corresponding matrix elements equal to one. The size of matrix of "Fourier-wavelet coefficients" is 512x512. So, different possible distributions of the root values of the generic algebraical systems (38) provide qualitatively different types of behaviour. Generic algebraic system (38), Generalized Dispersion Relation (GDR), provide the possibility for algebraic control. The above choice provides us by a distribution with chaotic-like equidistribution. But, if we consider a band-like structure of matrix (a_{ij}) with the band along the main diagonal with finite size ($\ll 512$) and values, e.g. five, while the other values are equal to one, we obtain localization in a fixed finite area of the full phase space, i.e. almost all energy of the system is concentrated in this small volume. This corresponds to wavelet states ⁷ and is shown in Fig. 9, constructed by means of Daubechies-based wavelet packets. Depending on the type of solution, such localization may be conserved during the whole time evolution (asymptotically-stable) or up to the needed value from the whole time scale (e.g. enough for plasma fusion/confinement in the case of fusion modeling by means of c-BBGKY hierarchy for dynamics of partitions).

7. Conclusions

By using wavelet bases with their best phase space localization properties, we can describe the localized (coherent) structures in quantum systems with complicated behaviour (Figs. 8, 11). The numerical simulation demonstrates the formation of different (stable) pattern or orbits generated by internal hidden symmetry from high-localized structures. Our (nonlinear) eigenmodes are more realistic for the modeling of nonlinear classical/quantum dynamical process than the corresponding linear gaussian-like coherent states. Here we mention only the best convergence properties of the expansions based on wavelet packets, which realize the minimal Shannon entropy property and the exponential control of convergence of expansions

like (39) based on the norm (25). Fig. 9 corresponds to (possible) result of superselection (einselection) [1] after decoherence process started from entangled state (Fig. 12); Fig. 10 and Fig. 13 demonstrate the steps of multiscale resolution (or degrees of interference) during modeling (quantum interaction/evolution) of entangled states leading to the growth of degree of entanglement. It should be noted that we can control the type of behaviour on the level of the reduced algebraical variational system, GDR (38).

Let us finish with some phenomenological description which can be considered as an attempt of qualitative description of the quantum dynamics as a whole and in comparison with its classical counterpart. It is possible to take for reminiscence the famous Dirac's phrase that "an electron can interact only itself via the process of quantum interference". Let G be the hidden/internal symmetry group on the spaces of quantum states which generates via MRA (22), (26) the multiscale/multiresolution representation for all dynamical quantities, unified in object $O(t)$, such as states, observables, partitions: $O^i(t) = \{\psi^i(t), Op^i(t), W_n^i(t)\}$, where i is the proper scale index. Then, the following commutative diagram represents the details of quantum life from the point of view of representations of G on the chosen functional realization which leads to decomposition of the whole quantum evolution into the proper orbits or scales corresponding to the proper level of resolution. Morphisms $W(t)$ describe Wigner-Weyl evolution in the algebra of symbols, while the processes of interactions with open World, such as the measurement or decoherence, correspond to morphisms (or even functors) $m(t)$ which transform the infinite set of scales characterizing the quantum object into finite ones, sometimes consisting of one element (demolition/destructive measurement).

$$\begin{array}{ccc}
 & W(t) & \\
 \{O^i(t_1)\} & \longrightarrow & \{O^j(t_2)\} \\
 \downarrow m(t_1) & & \downarrow m(t_2) \\
 & \widetilde{W(t)} & \\
 \{O^{i_c}(t_1)\} & \longrightarrow & \{O^{j_c}(t_2)\},
 \end{array}$$

where reduced morphisms $\widetilde{W(t)}$ correspond to (semi)classical or quasiclassical evolution. So, qualitatively,

Quantum Objects can be represented by an infinite or sufficiently large set of coexisting and interacting subsets like (22), (26) while

Classical Objects can be described by one or few only levels of resolution with (almost) suppressed interscale self-interaction. It is possible to consider Wigner functions as some measure of the quantum character of the system: as soon as it becomes positive, we arrive to classical regime and so there is no need to consider the full hierarchy decomposition in the representation (21).

So, Dirac's self-interference is nothing but the multiscale mixture/intermittency.

Certainly, the degree of this self-interaction leads to different qualitative types of behaviour, such as localized quasiclassical states, separable, entangled, chaotic etc. At the same time the instantaneous quantum interaction or transmission of (quantum) information from Alice to Bob takes place not in the physical kinematical space-time but in Hilbert spaces of states in their proper functional realization where there is a different kinematic life. To describe a set of Quantum Objects we need to realize our Space of States (Hilbert space) not as one functional space but as the so-called and well known in mathematics scale of spaces, e.g. $B_{p,q}^s, F_{p,q}^s$ ⁹. The proper multiscale decomposition for the scale of space provides us by the method of description of the set of quantum objects in case if the “size” of one Hilbert space of states is not enough to describe the complicated internal World. We will consider it elsewhere, while here we considered the one-scale case (to avoid possible misunderstanding we need to mention that one-scale case is also described by an infinite scale of spaces (26), but it is internal decomposition of the unique, attached to the problem, Hilbert space).

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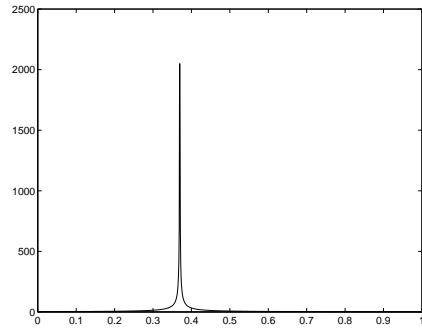


Fig. 2. Kick.

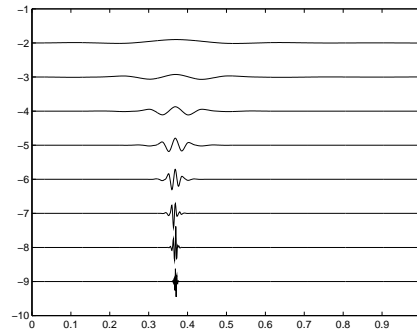


Fig. 5. MRA for Kick.

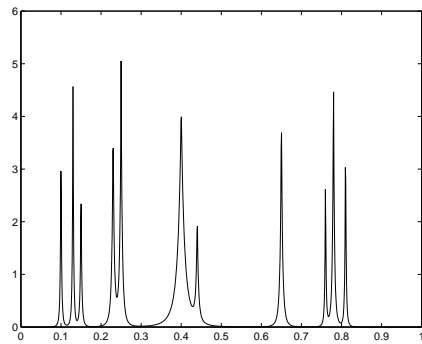


Fig. 3. Multi-Kicks.

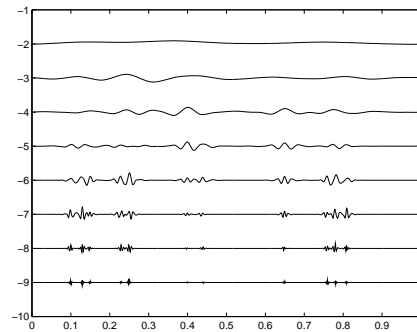


Fig. 6. MRA for Multi-Kicks.

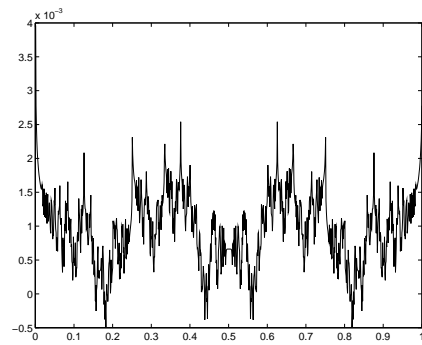


Fig. 4. RW-fractal.

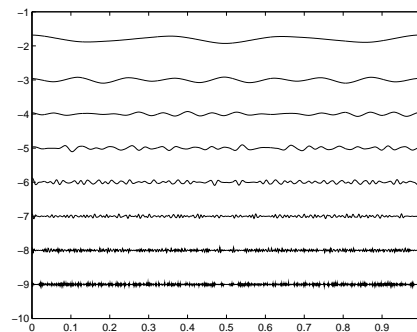


Fig. 7. MRA for RW-fractal.

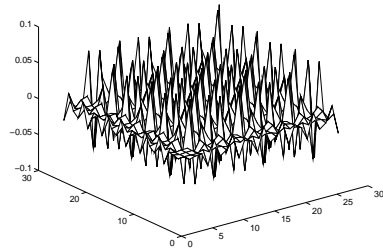


Fig. 8. Level 4 MRA.

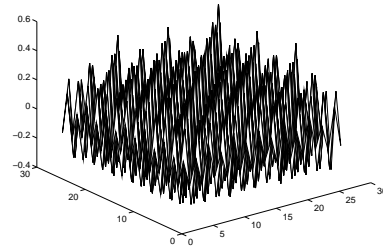


Fig. 11. Level 6 MRA.

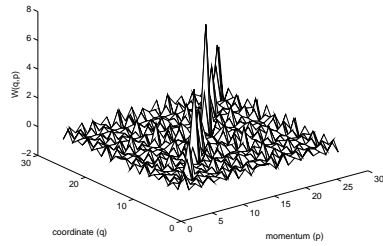


Fig. 9. Localized pattern, (wavelet) Wigner function.

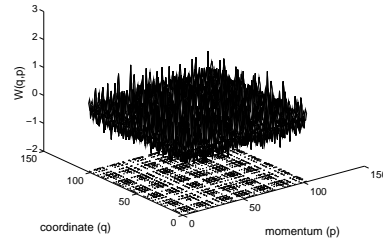


Fig. 12. Entangled-like Wigner function.

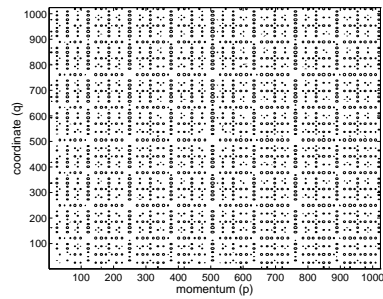


Fig. 10. Interference picture on the level 4 approximation for Wigner function.

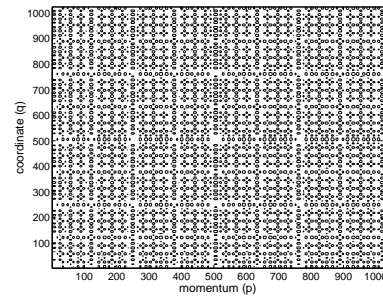


Fig. 13. Interference picture on the level 6 approximation for Wigner function.